Heat transfer in the thermal entrance region of a laminar non-Newtonian falling liquid film

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An analysis is presented for the entrance region heat transfer to a laminar, non-Newtonian falling liquid film with a fully developed velocity profile. Perturbation solutions are provided for the temperature field as well as heat transfer rates. The effect of heat generation by viscous dissipation is included. Numerical results for the heat transfer field are presented for a range of values of the power-law index and Brinkman number.

Keywords: falling liquid film; entrance region heat transfer; non-Newtonian liquid film

Introduction

There exist several industrial applications in which falling film heat exchangers are used widely. Non-Newtonian fluid falling film shell and tube exchangers are utilized in the food and polymer processing industries. In columns of small length, the falling film flow is laminar when the viscosity of the fluid is high.

Nusselt[1] predicted heat transfer to a vertical laminar film. Brauer[2] provided a relationship between wall shear and film heat transfer coefficients. Experimental data for film heating and evaporation were obtained by Ishigai *et al.*[3] The transient thermal response as well as conjugate heat transfer characteristics of a falling liquid film along a vertical surface were predicted by Gorla *et al.*[4,5] All these studies were concerned with Newtonian falling liquid films.

An integral approximate solution for the boundary layer equations in the case of a power-law type non-Newtonian laminar falling film was provided by Murthy and Sarma.[6] Heat transfer from an inclined plane to non-Newtonian fluid falling films was studied both theoretically and experimentally by Stucheli and Widmer.[7]

The present work has been undertaken in order to study the heat transfer in the thermal entrance region for an Ostwald–de Waele model power-law type of a non-Newtonian laminar falling film. The velocity field will be assumed to be fully developed, whereas the temperature field is taken as developing. The effect of heat generation by viscous dissipation is included in the analysis.

Analysis

Consideration may be given to a vertical plane placed in a parallel stream of a hydrodynamically fully developed non-Newtonian, laminar, falling liquid film. The liquid flow is characterized by the power-law rheological model. Let x and y denote the streamwise and normal coordinates, respectively. The total shear stress distribution in the liquid film is

$$\tau = K\left(\frac{du}{dy}\right)^n = \rho(\delta - y)g \tag{1}$$

where K is the consistency index and n is the power-law index.

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The shear stress at the wall is

$$\tau_w = K\left(\frac{du}{dy}\right)^n_{y=0} = \rho\,\delta g \tag{2}$$

The volumetric flow rate of liquid per unit width may be written as

$$\frac{Q}{B} = \int_0^\delta u\,dy \tag{3}$$

Using the boundary condition of no slip at the wall and zero interfacial shear at the gas-liquid interface, we may write

$$y=0:\quad u=0 \qquad y=\delta:\quad \frac{du}{dy}=0 \tag{4}$$

The velocity distribution is obtained by integrating Equation 1 and using the boundary conditions given by Equation 4. It may be written

$$\frac{u(\eta)}{U_0} = 1-(1-\eta)^{(n+1)/n} \tag{5}$$

where

$$\eta = y/\delta$$

$$U_0 = \left(\frac{n}{n+1}\right)\left(\frac{\rho g}{K}\right)^{1/n}\delta^{(n+1)/n}$$

Upon substituting Equation 5 into Equation 3, we obtain an expression for the film thickness:

$$\delta = \left\{\frac{[(2n+1)/n]Q}{B}\right\}^{n/(2n+1)} \Bigg/ \left(\frac{\rho g}{K}\right)^{1/(2n+1)} \tag{6}$$

The governing energy equation may be written as

$$u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{K}{\rho C_p}\left(\frac{du}{dy}\right)^{n+1} \tag{7}$$

with boundary conditions given by

$$x=0:\quad T=T_i \quad \text{(inlet condition)}$$
$$y=0:\quad T=T_w \tag{8}$$
$$y=\delta:\quad \frac{\partial T}{\partial y}=0 \quad \text{(zero interfacial heat flux)}$$

Proceeding with the analysis, one may define

$$\xi = x/L$$

$$\eta = y/\delta$$

$$\theta = \frac{T - T_w}{T_i - T_w}$$

$$\mathrm{Pe} = \frac{\rho C_p Q}{BK_f}$$

$$\mathrm{Gz} = \frac{\delta}{L}\,\mathrm{Pe}$$

$$\mathrm{Gz}^* = \mathrm{Gz}\left(\frac{2n+1}{n+1}\right)$$

$$\mathrm{Br} = \frac{K(Q/B)^{n+1}[(2n+1)/n]^{n+1}}{K_f(T_i - T_w)\delta^{2n}} \tag{9}$$

Substituting expressions in Equation 9 into Equations 7 and 8, we obtain the transformed energy equation

$$\mathrm{Gz}^*[1-(1-\eta)^{(n+1)/n}]\frac{d\theta}{d\xi} = \frac{\partial^2\theta}{\partial\eta^2} + \mathrm{Br}(1-\eta)^{(n+1)/n} \tag{10}$$

The transformed boundary conditions are

$$\begin{aligned} &\xi = 0: \quad \theta = 1 \\ &\eta = 0: \quad \theta = 0 \\ &\eta = 1, \quad \frac{\partial\theta}{\partial\eta} = 0 \end{aligned} \tag{11}$$

We may now define

$$\begin{aligned} &\phi = \eta[2\,\mathrm{Gz}^*/9\xi]^{1/3} \\ &\psi = (9\xi/2\,\mathrm{Gz}^*)^{1/3} \end{aligned} \tag{12}$$

Substituting Equation 12 into Equations 10 and 11, one may write

$$\frac{3}{2}\frac{1}{\psi^3}[1-(1-\phi\psi)^{(n+1)/n}]\left(-\phi\frac{\partial\theta}{\partial\phi} + \psi\frac{\partial\theta}{\partial\psi}\right) = \frac{1}{\psi^2}\frac{\partial^2\theta}{\partial\phi^2} + \mathrm{Br}(1-\phi\psi)^{(n+1)/n} \tag{13}$$

with boundary conditions

$$\begin{aligned} &\phi = 0: \quad \theta = 0 \\ &\phi \to \infty: \quad \theta \to 1 \end{aligned} \tag{14}$$

Solutions and results

A series solution will be sought for $\theta(\phi, \psi)$ in the form

$$\theta(\phi, \psi) = \theta_0(\phi) + \psi\theta_1(\phi) + \psi^2\theta_2(\phi) + \cdots \tag{15}$$

Substituting Equation 15 into Equations 13 and 14 and equating coefficients of like powers of ψ, we obtain

$$\theta_0'' + \frac{3(n+1)}{2n}\phi^2\theta_0' = 0 \tag{16}$$

$$\theta_1'' - \frac{3}{2}\left(\frac{n+1}{n}\right)\phi(\theta_1 - \phi\theta_1') - \frac{3}{2}\left(\frac{n+1}{2n}\right)\frac{1}{n}\phi^3\theta_0' = 0 \tag{17}$$

$$\theta_2'' - \frac{3}{2}\left(\frac{n+1}{n}\right)\phi(2\theta_2 - \phi\theta_2') + \frac{3}{2}\left(\frac{n+1}{2n}\right)\frac{1}{n}\phi^2(\theta_1 - \phi\theta_1') + \frac{1}{2}\left(\frac{n+1}{2n}\right)\frac{1}{n}\left(\frac{1}{n}-1\right)\phi^4\theta_0' + \mathrm{Br} = 0 \tag{18}$$

$$\theta_3'' - \frac{3}{2}\left(\frac{n+1}{n}\right)\phi(3\theta_3 - \phi\theta_3') + \frac{3}{2}\left(\frac{n+1}{2n}\right)\frac{1}{n}\phi^2(2\theta_2 - \phi\theta_2') - \frac{1}{2}\left(\frac{n+1}{2n}\right)\frac{1}{n}\left(\frac{1}{n}-1\right)(\theta_1 - \phi\theta_1')\phi^3 - \frac{1}{8}\left(\frac{n+1}{2n}\right)\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\phi^5\theta_0' - \mathrm{Br}\left(\frac{n+1}{n}\right)\phi = 0 \tag{19}$$

$$\theta_4'' - \frac{3}{2}\left(\frac{n+1}{n}\right)\phi(4\theta_4 - \phi\theta_4') + \frac{3}{2}\left(\frac{n+1}{2n}\right)\frac{1}{n}\phi^2(3\theta_3 - \phi\theta_3') - \frac{1}{2}\left(\frac{n+1}{2n}\right)\frac{1}{n}\left(\frac{1}{n}-1\right)\phi^3(2\theta_2 - \phi\theta_2') + \frac{1}{8}\left(\frac{n+1}{2n}\right)\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\phi^4(\theta_1 - \phi\theta_1') + \frac{1}{40}\left(\frac{n+1}{2n}\right)\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\left(\frac{1}{n}-3\right)\phi^6\theta_0' + \frac{\mathrm{Br}}{2}\left(\frac{n+1}{n}\right)\frac{1}{n}\phi^2 = 0 \tag{20}$$

etc.

Notation

B	Width of plate
Br	Brinkman number
C_p	Specific heat
g	Gravitational acceleration
Gz	Graetz number
Gz*	Modified Graetz number
h	Heat transfer coefficient
$\bar{h}$	Mean heat transfer coefficient
K	Viscosity coefficient for power-law fluids
K_f	Thermal conductivity
L	Characteristic length
n	Power-law index
Nu	Nusselt number
Pe	Peclet number
Q	Volume flow rate
T	Temperature
x	Coordinate in the flow direction
y	Normal coordinate
ξ	Dimensionless axial coordinate
α	Thermal diffusivity
η	Dimensionless normal coordinate
δ	Film thickness
θ	Dimensionless temperature
ρ	Density
ϕ	Transformed dimensionless normal coordinate
ψ	Transformed dimensionless axial coordinate

Subscripts

i	Inlet conditions
w	Surface conditions

Superscript

$\bar{}$	Average condition

The primes denote differentiation with respect to ϕ only. The governing boundary conditions are

$$\theta_0(0)=\theta_1(0)=\theta_2(0)=\cdots=0 \qquad \theta_0(\infty)=1, \quad \theta_1(\infty)=\theta_2(\infty)=\cdots=0 \tag{21}$$

Equation 16 under boundary conditions $\theta_0(0)=0$ and $\theta_0(\infty)=1$ can be solved in a closed form. The solution is

$$\theta_0=\frac{((n+1)/2n)^{1/3}\theta \int_0 e^{-\beta^2}\,d\beta}{\Gamma(4/3)} \tag{22}$$

Equations 17–20 were not solved in closed form but have been integrated numerically. The distribution of the thermal functions θ_0, θ_1, θ_2, θ_3, and θ_4 is illustrated in Figures 1–3. Here, n and Br are treated as prescribable parameters. The dimensionless temperature may be subsequently obtained by means of Equation 15.

The local heat flux at the wall is

$$q_w=-K_f\left(\frac{\partial T}{\partial y}\right)_{y=0} = -\frac{K_f(T_i-T_w)}{\delta}\left(\frac{2Gz^*}{9\xi}\right)^{1/3}\left(\frac{\partial\theta}{\partial\phi}\right)_{\phi=0} \tag{23}$$

The local heat transfer coefficient may be defined based upon the inlet temperature difference as

$$h(x)=\frac{q_w}{T_w-T_i} \tag{24}$$

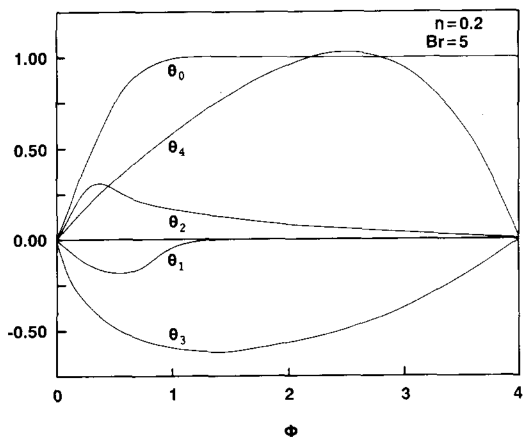

Figure 1 Temperature distribution functions ($n=0.2$, Br$=5$)

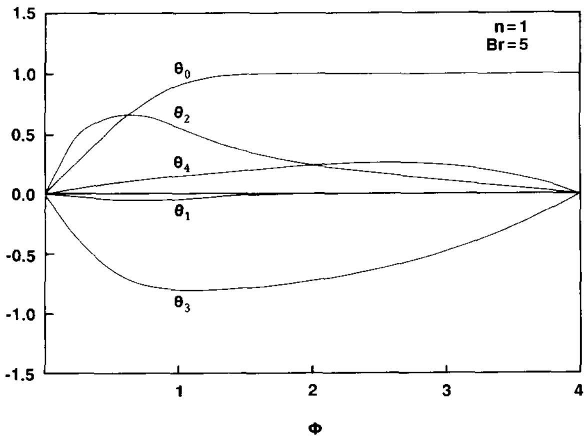

Figure 2 Temperature distribution functions ($n=1$, Br$=5$)

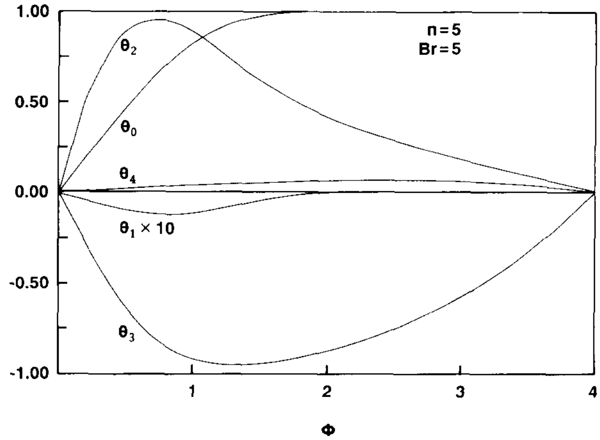

Figure 3 Temperature distribution functions ($n=5$, Br$=5$)

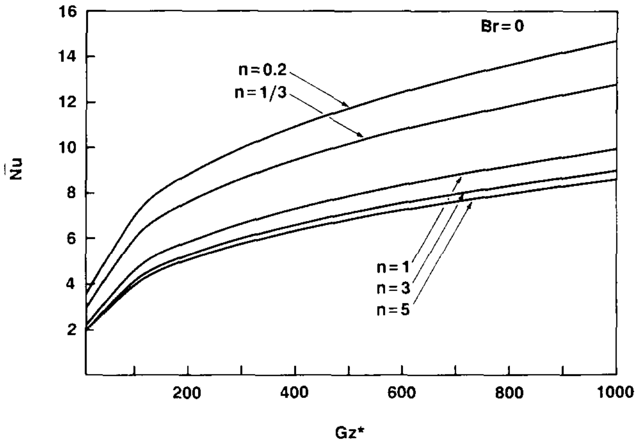

Figure 4 Average Nusselt number versus the modified Graetz number (Br$=0$)

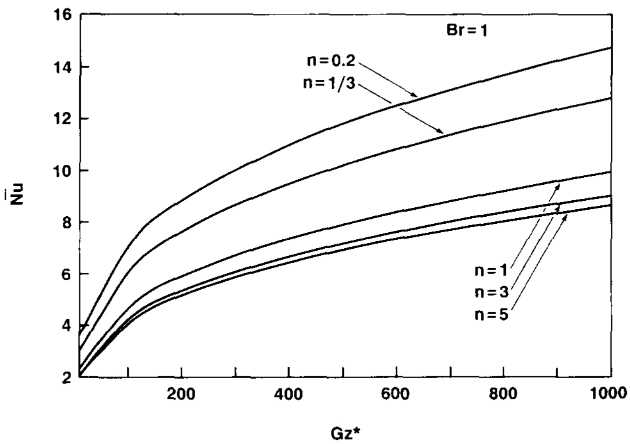

Figure 5 Average Nusselt number versus the modified Graetz number (Br$=1$)

The local Nusselt number therefore becomes

$$Nu_x=\frac{hx}{K_f} = \frac{L}{\delta}\xi\left(\frac{2Gz^*}{9\xi}\right)^{1/3}[\theta_0'+\psi\theta_1'(0)+\psi^2\theta_2'(0)+\cdots] \tag{25}$$

Table 1 Values of θ'_j versus n and Br (j=0, 1, 2, 3, and 4)

n	Br	$\theta'_0(0)$	$\theta'_1(0)$	$\theta'_2(0)$	$\theta'_3(0)$	$\theta'_4(0)$
	0	1.615095	−0.485754	−0.033408	0.017077	0.021368
0.2	1	1.615095	−0.485754	0.316650	−0.385672	0.197560
	5	1.615095	−0.485754	1.716942	−1.996666	0.902385
	0	1.410913	−0.292540	−0.032753	−0.001699	0.003904
1/3	1	1.410913	−0.292540	0.367731	−0.364810	0.114292
	5	1.410913	−0.292540	1.969725	−1.817137	0.556022
	0	1.119852	−0.097990	−0.017911	−0.005633	−0.002235
1	1	1.119852	−0.097990	0.485748	−0.320226	0.039250
	5	1.119852	−0.097990	2.500623	−1.578480	0.205010
	0	0.978231	−0.032854	−0.007421	−0.002831	−0.001401
3	1	0.978231	−0.032854	0.568181	−0.300199	0.013977
	5	0.978231	−0.032854	2.870649	−1.489669	0.075430
	0	0.944471	−0.019693	−0.004679	−0.001878	−0.000983
5	1	0.944471	−0.019693	0.591189	−0.296146	0.008553
	5	0.944471	−0.019693	2.974540	−1.472801	0.046819

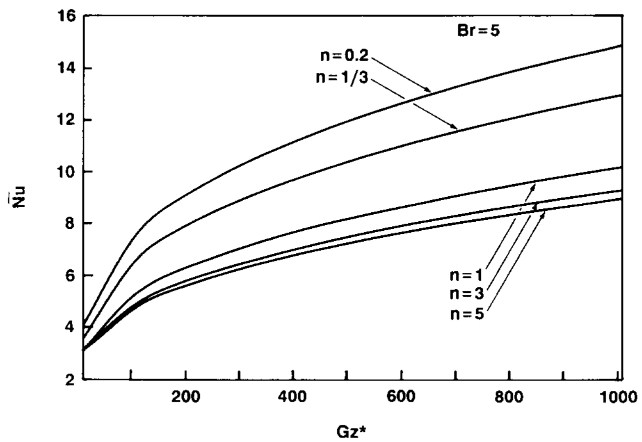

Figure 6 Average Nusselt number versus the modified Graetz number (Br=5)

If $\bar{h}$ is the mean heat transfer coefficient for heat transfer based on the inlet temperature difference, one may write

$$L\bar{h}(T_w - T_i) = -K_f \int_0^L \left(\frac{\partial T}{\partial y}\right)_{y=0} dx \tag{26}$$

The average Nusselt number therefore becomes

$$\overline{\mathrm{Nu}} = \left(\frac{2\mathrm{Gz}^*}{9}\right)^{1/3} \int_0^1 \xi^{-1/3}\left[\left(\frac{\partial\theta}{\partial\phi}\right)_{\phi=0}\right] d\xi \tag{27}$$

In most practical applications, it is the surface characteristics, such as heat transfer rate, which are important. The values of $\theta'_0(0)$, $\theta'_1(0)$, $\theta'_2(0)$, $\theta'_3(0)$, and $\theta'_4(0)$ which are proportional to the Nusselt number have been tabulated in Table 1. With this information one may compute the local Nusselt number or average Nusselt number in a straightforward way from Equations 25 and 27, respectively. The variation of the average Nusselt number versus the modified Graetz number is shown in Figures 4–6. In these figures, the power-law index n and the Brinkman number have been treated as prescribable parameters. The results indicate that the effect of viscous dissipation is to increase the average heat transfer rate. For a given value of the Graetz number, the average dimensionless heat transfer rate is higher in pseudoplastic fluids ($n<1$) than in dilatant ($n>1$) fluids. For a given fluid, in general, the average heat transfer rate increases with the modified Graetz number.

Concluding remarks

The heat transfer in the thermal entrance region of an Ostwald–de Waele type non-Newtonian laminar falling film has been investigated. Perturbation solutions are provided for the temperature distribution within the film as well as for the heat transfer to the film. The results indicate that the average Nusselt number increases with the modified Graetz number. For a given value of the modified Graetz number, the average heat transfer rate was found to be greater for pseudoplastic fluids than for dilatant fluids. The effect of heat generation is to augment the average heat transfer rate.

Both θ_0 and θ_1 are independent of the Brinkman number. Therefore for large values of the Graetz number, heat generation by viscous dissipation becomes less important. Unless Br is at least $(\mathrm{Gz}^*)^{2/3}$, the heat generation by viscous dissipation does not become important.

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